NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3229

THE SMALL-DISTURBANCE METHOD FOR FLOW OF A COMPRESSIBLE FLUID WITH VELOCITY POTENTIAL AND STREAM FUNCTION AS INDEPENDENT VARIABLES

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SUMMARY

The equations of two-dimensional compressible flow are treated according to the Prandtl-Busemann small-disturbance method. In contrast to the usual procedure, the independent variables are the compressible velocity potential and stream function and the dependent variables are the rectangular Cartesian coordinates in the plane of flow. The six first-order differential equations corresponding to the first three iteration steps are put into complex-vector form. The particular integrals of the resulting set of three equations are then directly obtained. As an example, the general results of the analysis are applied to the case of subsonic compressible flow past a sinusoidal wall of small amplitude.

INTRODUCTION

The problem of the integration of the equations of compressible flow past a prescribed solid boundary has been treated in numerous papers and in several diverse ways. With the exception of the hodograph-transformation method, the equations are usually solved by methods of successive approximations. One of these, initiated by Janzen and Rayleigh, starts from the incompressible-flow solution and develops the compressibility effects in a series of powers of the undisturbed stream Mach number. This method is restricted to the subsonic range, since the differential equations of the process are always of the elliptic type. In general, this method yields the best results for flows past thick shapes for which the critical stream Mach number is much less than unity. In contrast, the Prandtl-Busemann small-disturbance method of iteration in terms of a small parameter starts from the undisturbed flow, the first step being the well-known Prandtl linearized solution. This method is best suited to thin profiles for which the critical stream Mach number is nearly equal to unity and, from the beginning, yields a good approximation to the desired rigorous solution. Both the Janzen-Rayleigh and the Prandtl-Busemann methods are applicable not only to plane but also to axisymmetric subsonic flows. The Prandtl-Busemann procedure has also been utilized for the calculation of plane and axisymmetric supersonic flows. A good discussion of these two iterative procedures, including a number of basic references, is given in reference 1.

The author has recently considered the problem of plane incompressible flow past an infinitely long, sinusoidal wall of arbitrary amplitude (ref. 2). It was found that this problem could not be treated in the physical-flow plane but had to be transferred to the plane of velocity potential and stream function. In this plane, the problem was not only successfully treated by the small-disturbance method but also, the solution was rigorously expressed in the form of a nonlinear integral equation. This unexpected result has prompted the present investigation of the form of the Prandtl-Busemann small-disturbance equations with the velocity potential and the stream function as independent variables. The equation of continuity and the condition of irrotationality form the basis of the procedure. Each iteration step consists then of a pair of first-order differential equations of the Cauchy-Riemann type rather than a single second-order differential equation of the Laplace or Poisson type. These pairs of equations, moreover, can be put into complex-vector forms and their particular integrals obtained in a straightforward manner. The present paper contains the results of this investigation inclusive of the third-order step. The example of subsonic flow past a sinusoidal wall of small amplitude is utilized to illustrate the general results of the analysis. No counterpart of the integral-equation approach of reference 1 could be found.

ANALYSIS

The requirements of continuity and irrotationality lead to the relations

$$\left.\begin{aligned} u &= \phi_x = \frac{\rho_\infty}{\rho}\,\psi_y \\ v &= \phi_y = -\frac{\rho_\infty}{\rho}\,\psi_x \end{aligned}\right\} \qquad (1)$$

where

u,v velocity components in direction of x- and y-axes, respectively

x,y rectangular Cartesian coordinates in flow plane

ρ density of fluid

ρ_∞ density of fluid in undisturbed flow

ϕ velocity potential

ψ stream function

The subscripts denote differentiation with respect to the designated variables. The quantities x, y, ϕ, ψ, u, and v are nondimensional with a characteristic length associated with the solid boundary as unit of length and the speed U of the undisturbed stream as unit of velocity.

From Bernoulli's theorem there follows the relation

$$\frac{\rho}{\rho_\infty} = \left[1 - \frac{\gamma - 1}{2} M_\infty^2(q^2 - 1)\right]^{\frac{1}{\gamma-1}} \tag{2}$$

where

M_∞ Mach number of undisturbed stream

q nondimensional speed of fluid

γ ratio of specific heats at constant pressure and constant volume

The introduction of ϕ and ψ as independent variables leads to the following relations in place of equations (1):

$$\left.\begin{aligned} y_\psi &= \frac{\rho_\infty}{\rho} x_\phi \\ x_\psi &= -\frac{\rho_\infty}{\rho} y_\phi \end{aligned}\right\} \tag{3}$$

For the purpose of deriving the iteration equations of the small-disturbance method, let

$$\left.\begin{aligned} x &= \phi + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \ldots \\ y &= \psi + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \ldots \end{aligned}\right\} \tag{4}$$

where x_n and y_n $(n = 1, 2, 3 \ldots)$ are functions of the independent variables ϕ and ψ and where ϵ is a suitable small parameter which serves to regulate the course of the iteration process.

Now,

$$q^2 = \frac{{x_\psi}^2 + {y_\psi}^2}{(x_\phi y_\psi - x_\psi y_\phi)^2} \tag{5}$$

and

$$\left.\begin{aligned}
x_\phi &= 1 + \epsilon x_{1\phi} + \epsilon^2 x_{2\phi} + \epsilon^3 x_{3\phi} + \ldots \\
x_\psi &= \epsilon x_{1\psi} + \epsilon^2 x_{2\psi} + \epsilon^3 x_{3\psi} + \ldots \\
y_\phi &= \epsilon y_{1\phi} + \epsilon^2 y_{2\phi} + \epsilon^3 y_{3\phi} + \ldots \\
y_\psi &= 1 + \epsilon y_{1\psi} + \epsilon^2 y_{2\psi} + \epsilon^3 y_{3\psi} + \ldots
\end{aligned}\right\} \tag{6}$$

Substituting equations (6) into equation (5) yields

$$\begin{aligned}
q^2 - 1 = -2\epsilon x_{1\phi} + \epsilon^2\left(3{x_{1\phi}}^2 + 2x_{1\psi}y_{1\phi} + {x_{1\psi}}^2 - 2x_{2\phi}\right) + 2\epsilon^3\Big(-2{x_{1\phi}}^3 - \\
3x_{1\phi}x_{1\psi}y_{1\phi} - x_{1\psi}y_{1\phi}y_{1\psi} + 3x_{1\phi}x_{2\phi} + x_{1\psi}y_{2\phi} + x_{2\psi}y_{1\phi} + \\
x_{1\psi}x_{2\psi} - x_{1\phi}{x_{1\psi}}^2 - {x_{1\psi}}^2 y_{1\psi} - x_{3\phi}\Big) + \ldots
\end{aligned} \tag{7}$$

Also, from equation (2),

$$\begin{aligned}
\frac{\rho}{\rho_\infty} = 1 - \frac{1}{2} {M_\infty}^2(q^2 - 1) + \frac{2 - \gamma}{8} {M_\infty}^4(q^2 - 1)^2 - \\
\frac{(2 - \gamma)(3 - 2\gamma)}{48} {M_\infty}^6(q^2 - 1)^3 + \ldots
\end{aligned} \tag{8}$$

By means of equations (5), (6), (7), and (8), equations (3) yield the following sets of relations when the coefficients of the various powers of ϵ are equated to zero:

$$\left.\begin{aligned} \beta^2 x_{1\phi} &= y_{1\psi} \\ x_{1\psi} &= -y_{1\phi} \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} \beta^2 x_{2\phi} &= y_{2\psi} + \frac{1}{2} M_\infty^2 \left\{ y_{1\phi}^2 - \left[\beta^2 + (\gamma + 1)M_\infty^2\right] x_{1\phi}^2 \right\} \\ x_{2\psi} &= -y_{2\phi} + M_\infty^2 x_{1\phi} y_{1\phi} \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} \beta^2 x_{3\phi} = y_{3\psi} + M_\infty^2 \Bigg\{ & y_{1\phi} y_{2\phi} - \left(1 + \frac{1}{2}\beta^2 + \frac{\gamma + 1}{2} M_\infty^2\right) x_{1\phi} y_{1\phi}^2 - \\ & \left[\beta^2 + (\gamma + 1)M_\infty^2\right] x_{1\phi} x_{2\phi} + \Bigg[\frac{1}{2}\beta^4 + (\gamma + 1)M_\infty^2 - \\ & \frac{5}{6}(\gamma + 1)M_\infty^4 + \frac{1}{3}(\gamma + 1)^2 M_\infty^4\Bigg] x_{1\phi}^3 \Bigg\} \\ x_{3\psi} = -y_{3\phi} + M_\infty^2 \Bigg[& \frac{1}{2} y_{1\phi}^3 + x_{2\phi} y_{1\phi} + x_{1\phi} y_{2\phi} - \\ & \left(1 + \frac{1}{2}\beta^2 + \frac{\gamma + 1}{2} M_\infty^2\right) x_{1\phi}^2 y_{1\phi} \Bigg] \end{aligned}\right\} \tag{11}$$

where $\beta^2 = 1 - M_\infty^2$.

By defining

$$x_n + \frac{i}{\beta} y_n = z_n \qquad x_n - \frac{i}{\beta} y_n = \bar{z}_n$$

$$\phi + i\beta\psi = w \qquad \phi - i\beta\psi = \bar{w}$$

and making use of the symbolic relations

$$\frac{\partial}{\partial \phi} = \frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}}$$

$$\frac{\partial}{\partial \psi} = i\beta\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial \bar{w}}\right)$$

equations (9), (10), and (11) can be expressed in the following complex-vector forms:

$$z_{1\bar{w}} = 0 \tag{12}$$

$$z_{2\bar{w}} = -\frac{\gamma + 1}{8}\frac{M_\infty^4}{\beta^2}\left(\frac{1}{2} z_{1w}^2 + z_{1w}\bar{z}_{1\bar{w}}\right) - \frac{1}{4} M_\infty^2\left(1 + \frac{\gamma + 1}{4}\frac{M_\infty^2}{\beta^2}\right)\bar{z}_{1\bar{w}}^2 \tag{13}$$

$$\begin{aligned} z_{3\bar{w}} = {} & -\frac{1}{2} M_\infty^2 \bar{z}_{1\bar{w}}\left(\bar{z}_{2\bar{w}} + \bar{z}_{2w}\right) - \frac{1}{8} M_\infty^2\left(1 + \frac{1}{2}\beta^2 + \frac{\gamma + 1}{2} M_\infty^2\right)\bar{z}_{1\bar{w}}\left(z_{1w}^2 - \bar{z}_{1\bar{w}}^2\right) - \\ & \frac{1}{32} M_\infty^2\beta^2\left(z_{1w} - \bar{z}_{1\bar{w}}\right)^3 - \frac{1}{8}(\gamma + 1)\frac{M_\infty^4}{\beta^2}\left(z_{1w} + \bar{z}_{1\bar{w}}\right)\left(z_{2w} + z_{2\bar{w}} + \bar{z}_{2w} + \right. \\ & \left.\bar{z}_{2\bar{w}}\right) + \frac{1}{16} M_\infty^2\left[\frac{1}{2}\beta^2 + (\gamma + 1)\frac{M_\infty^2}{\beta^2} - \frac{5}{6}(\gamma + 1)\frac{M_\infty^4}{\beta^2} + \right. \\ & \left.\frac{1}{3}(\gamma + 1)^2\frac{M_\infty^4}{\beta^2}\right]\left(z_{1w} + \bar{z}_{1\bar{w}}\right)^3 \end{aligned} \tag{14}$$

Corresponding to these equations are the following second-order partial-differential equations:

$$z_{1w\bar{w}} = 0 \tag{15}$$

$$z_{2w\bar{w}} = -\frac{\gamma + 1}{4}\frac{M_\infty^{\ 4}}{\beta^2} x_{1\phi} z_{1\phi\phi} \tag{16}$$

$$z_{3w\bar{w}} = -\frac{\gamma + 1}{4}\frac{M_\infty^{\ 4}}{\beta^2}\left(\left\{\frac{1}{2} y_{1\phi}^{\ 2} + x_{2\phi} - \left[\frac{3}{2} + \beta^2 + (\gamma + 1)M_\infty^{\ 2}\right]x_{1\phi}^{\ 2}\right\}z_{1\phi\phi} + x_{1\phi}z_{2\phi\phi}\right) \tag{17}$$

Note that, when ϕ and ψ are the independent variables, the basic iteration equations (12), (13), and (14) for x and y are of the first order. In contrast, when x and y are the independent variables, the basic iteration equations for ϕ or ψ are of the second order and are similar to equations (15), (16), and (17).

CALCULATION OF PARTICULAR INTEGRALS

Equations (12), (13), and (14) can be integrated without difficulty. The general solutions are then obtained by the addition of arbitrary functions of w to the particular integrals. These arbitrary functions are of the nature of constants of integration and are chosen in such a way that the boundary conditions of the problem under consideration are satisfied.

Both the particular and the general solution of equation (12) therefore is simply that z_1 is a function of w only. A particular integral of equation (13) can then be written as follows:

$$z_2 = -\frac{\gamma + 1}{8}\frac{M_\infty^{\ 4}}{\beta^2}\left[\frac{1}{2}(\bar{w} - w)z_{1w}^{\ 2} + \left(z_1 + \bar{z}_1\right)z_{1w}\right] - \frac{1}{4}M_\infty^{\ 2}\left(1 + \frac{\gamma + 1}{4}\frac{M_\infty^{\ 2}}{\beta^2}\right)\int \bar{z}_{1\bar{w}}^{\ 2}d\bar{w} \tag{18}$$

By substituting this expression for z_2 into equation (14), the following particular integral can be obtained:

$$z_3 = \frac{1}{8} M_\infty^2\left(1 + \beta^2\right)\int \bar{z}_{1\bar{w}}^{\,3} d\bar{w} + \frac{\gamma + 1}{32} \frac{M_\infty^4}{\beta^2}\left\{M_\infty^2\left(1 + \frac{\gamma + 1}{4} \frac{M_\infty^2}{\beta^2}\right)\left[z_1 + \bar{z}_1 + \right.\right.$$

$$\left.\frac{2}{3} (w - \bar{w})\bar{z}_{1\bar{w}}\right]\bar{z}_{1\bar{w}}^{\,2} + \frac{2}{3}\left[2 + \beta^2 + (\gamma + 1) \frac{M_\infty^2}{\beta^2} + \right.$$

$$\left.\frac{1}{8} (\gamma + 1) \frac{M_\infty^4}{\beta^2}\right](\bar{w} - w)z_{1w}^{\,3} + \left[3 + \beta^2 + 2(\gamma + 1) \frac{M_\infty^2}{\beta^2} - \right.$$

$$\left.\frac{1}{4} (\gamma + 1) \frac{M_\infty^4}{\beta^2}\right]\bar{z}_1 z_{1w}^{\,2} + 2\left[2 + \beta^2 + (\gamma + 1) \frac{M_\infty^2}{\beta^2} - \right.$$

$$\left.\left.\frac{1}{4} (\gamma + 1) \frac{M_\infty^4}{\beta^2}\right]z_{1w}\int \bar{z}_{1\bar{w}}^{\,2} d\bar{w} + \left[3 + \beta^2 + \frac{2}{3} (\gamma + 1) \frac{M_\infty^2}{\beta^2}\right]\int \bar{z}_{1\bar{w}}^{\,3} d\bar{w}\right\} +$$

$$\left(\frac{\gamma + 1}{8} \frac{M_\infty^4}{\beta^2}\right)^2 \left\{\frac{1}{2}\left[z_1 + \bar{z}_1 + (\bar{w} - w)z_{1w}\right]^2 z_{1ww} + \left[z_1 + \bar{z}_1 + \right.\right.$$

$$\left.\left.\frac{1}{2} (w - \bar{w})\bar{z}_{1\bar{w}}\right]z_{1w}\bar{z}_{1\bar{w}}\right\} \tag{19}$$

The analysis thus far has not been restricted and applies equally well to both subsonic and supersonic flows. Actually, of course, x_1 and y_1 as functions of ϕ and ψ are radically different for the two types of flow and $\beta^2 = M_\infty^{\ 2} - 1$ in the case of supersonic flow. In order to illustrate the application of the general results to a particular flow problem, the following section contains a treatment of the problem of subsonic compressible flow past a sinusoidal wall of small amplitude.

SUBSONIC FLOW PAST A SINUSOIDAL WALL

The equation of the solid boundary is

$$y = \epsilon \cos x \tag{20}$$

and the flow is assumed to be in the direction of the positive x-axis. The undisturbed stream velocity U and the wave length λ in radians are utilized as units of velocity and length, respectively. The parameter of the problem is ϵ, the amplitude of the wavy wall in radians. The appropriate solution of equation (12), vanishing for $\psi = \infty$, is simply

$$z_1 = iAe^{iw}$$

where A is a real quantity to be determined by the boundary condition at the surface, namely, $\psi = 0$ and $y = \epsilon \cos x$. Then

$$x = \phi - \epsilon A \sin \phi$$

and

$$\cos x = \beta A \cos \phi$$

To the zeroeth order of ϵ, $\cos x \approx \cos \phi$ and, therefore, $A = \frac{1}{\beta}$, or

$$\left.\begin{aligned} x &= \phi - \frac{\epsilon}{\beta} e^{-\beta\psi} \sin \phi \\ \frac{1}{\beta} y &= \frac{1}{\beta} \psi + \frac{\epsilon}{\beta} e^{-\beta\psi} \cos \phi \end{aligned}\right\} \tag{21}$$

With $z_1 = \frac{i}{\beta} e^{iw}$, the appropriate general expression for z_2 can be obtained from equation (18) and is

$$z_2 = i \frac{\gamma + 1}{8} \frac{M_\infty{}^4}{\beta^4} \left[(\beta\psi + 1)e^{2iw} - e^{i(w-\overline{w})}\right] -$$

$$\frac{i}{8} \frac{M_\infty{}^2}{\beta^2}\left(1 + \frac{\gamma + 1}{4} \frac{M_\infty{}^2}{\beta^2}\right)e^{-2i\overline{w}} + iBe^{2iw} + iC$$

where B and C are real quantities to be determined by the boundary condition at the surface, that is, $\psi = 0$ with $y = \epsilon \cos x$. Therefore,

$$x = \phi - \frac{\epsilon}{\beta} \sin \phi - \left(\frac{\epsilon}{\beta}\right)^2 \left[\frac{1}{8} M_\infty{}^2\left(1 + 5 \frac{\gamma + 1}{4} \frac{M_\infty{}^2}{\beta^2}\right) + \beta^2 B\right] \sin 2\phi \tag{22a}$$

and

$$\cos x = \cos \phi + \frac{\epsilon}{\beta} \left\{- \frac{1}{8} M_\infty{}^2 \left[\cos 2\phi + (\gamma + 1) \frac{M_\infty{}^2}{\beta^2} \left(1 - \frac{3}{4} \cos 2\phi\right)\right] + \beta^2 B \cos 2\phi + \beta^2 C\right\} \tag{22b}$$

From equation (22a), to the first order of ϵ/β,

$$\cos x \approx \cos \phi + \frac{1}{2} \frac{\epsilon}{\beta} (1 - \cos 2\phi)$$

Hence, comparison with the right-hand side of equation (22b) shows that

$$\beta^2 B = -\frac{1}{2} + \frac{1}{8} M_\infty{}^2 - \frac{3}{32} (\gamma + 1) \frac{M_\infty{}^4}{\beta^2}$$

and

$$\beta^2 C = \frac{1}{2} + \frac{\gamma + 1}{8} \frac{M_\infty{}^4}{\beta^2}$$

Thus, to the second order of ϵ/β,

$$x = \phi - \frac{\epsilon}{\beta} e^{-\beta\psi} \sin \phi + \frac{1}{4}\left(\frac{\epsilon}{\beta}\right)^2 \left[1 + \beta^2 - \frac{\gamma + 1}{4} \frac{M_\infty{}^4}{\beta^2} (2\beta\psi + 1)\right] e^{-2\beta\psi} \sin 2\phi \tag{23a}$$

and

$$\frac{1}{\beta} y = \frac{1}{\beta} \psi + \frac{\epsilon}{\beta} e^{-\beta\psi} \cos \phi + \frac{1}{2}\left(\frac{\epsilon}{\beta}\right)^2 \left\{1 + \frac{\gamma + 1}{4} \frac{M_\infty{}^4}{\beta^2} - e^{-2\beta\psi}\left[\frac{\gamma + 1}{4} \frac{M_\infty{}^4}{\beta^2} + \left(1 - \frac{\gamma + 1}{4} \frac{M_\infty{}^4}{\beta^2} \beta\psi\right) \cos 2\phi\right]\right\} \tag{23b}$$

Again, with $z_1 = \frac{i}{\beta} e^{iw}$, equation (19) leads to the general expression for z_3:

$$z_3 = -\frac{i}{24}\frac{{M_\infty}^2}{\beta^3}(1 + \beta^2)e^{-3i\overline{w}} + i\,\frac{\gamma + 1}{32}\frac{{M_\infty}^4}{\beta^5}\left\{{M_\infty}^2\left(1 + \frac{\gamma + 1}{4}\frac{{M_\infty}^2}{\beta^2}\right)\left[e^{i(w-2\overline{w})} - \right.\right.$$

$$\left.\left(1 + \frac{4}{3}\beta\psi\right)e^{-3i\overline{w}}\right] + \frac{4}{3}\left[2 + \beta^2 + (\gamma + 1)\frac{{M_\infty}^2}{\beta^2} + \frac{1}{8}(\gamma + 1)\frac{{M_\infty}^4}{\beta^2}\right]\beta\psi e^{3iw} -$$

$$\left[3 + \beta^2 + 2(\gamma + 1)\frac{{M_\infty}^2}{\beta^2} - \frac{1}{4}(\gamma + 1)\frac{{M_\infty}^4}{\beta^2}\right]e^{i(2w-\overline{w})} - \left[2 + \beta^2 + \right.$$

$$\left.(\gamma + 1)\frac{{M_\infty}^2}{\beta^2} - \frac{1}{4}(\gamma + 1)\frac{{M_\infty}^4}{\beta^2}\right]e^{i(w-2\overline{w})} - \left[1 + \frac{1}{3}\beta^2 + \right.$$

$$\left.\frac{2}{9}(\gamma + 1)\frac{{M_\infty}^2}{\beta^2}\right]e^{-3i\overline{w}} + \frac{\gamma + 1}{4}\frac{{M_\infty}^4}{\beta^2}\left[(1 + 2\beta\psi)^2 e^{3iw} - 4\beta\psi e^{i(2w-\overline{w})} - \right.$$

$$\left.\left.(1 + 2\beta\psi)e^{i(w-2\overline{w})}\right]\right\} + iDe^{iw} + iEe^{3iw} \tag{24}$$

where D and E are real quantities to be determined by the boundary condition at the surface, that is, $\psi = 0$ with $y = \epsilon \cos x$. Therefore,

$$x = \phi - \frac{\epsilon}{\beta}\sin\phi + \frac{1}{4}\left(\frac{\epsilon}{\beta}\right)^2\left(1 + \beta^2 - \frac{\gamma + 1}{4}\frac{{M_\infty}^4}{\beta^2}\right)\sin 2\phi + \mathrm{R.P.}z_3 \tag{25a}$$

and

$$\cos x = \cos \phi + \frac{1}{2}\frac{\epsilon}{\beta}(1 - \cos 2\phi) + \left(\frac{\epsilon}{\beta}\right)^2\left(-\frac{1}{24}M_\infty^{\ 2}(1 + \beta^2)\cos 3\phi + \frac{\gamma + 1}{32}\frac{M_\infty^{\ 4}}{\beta^2}\left\{\left[-4 - 3\beta^2 - 3(\gamma + 1)\frac{M_\infty^{\ 2}}{\beta^2} + \frac{1}{2}(\gamma + 1)\frac{M_\infty^{\ 4}}{\beta^2}\right]\cos \phi + \left[-2 + \frac{2}{3}\beta^2 - \frac{2}{9}(\gamma + 1)\frac{M_\infty^{\ 2}}{\beta^2}\right]\cos 3\phi\right\} + \beta^3 D \cos \phi + \beta^3 E \cos 3\phi\right) \tag{25b}$$

Now, from equation (25a), to the second order of ϵ/β,

$$\cos x \approx \frac{1}{2}\frac{\epsilon}{\beta} + \left[1 - \frac{1}{8}\left(\frac{\epsilon}{\beta}\right)^2\left(2 + \beta^2 - \frac{\gamma + 1}{4}\frac{M_\infty^{\ 4}}{\beta^2}\right)\right]\cos \phi - \frac{1}{2}\frac{\epsilon}{\beta}\cos 2\phi + \frac{1}{8}\left(\frac{\epsilon}{\beta}\right)^2\left(2 + \beta^2 - \frac{\gamma + 1}{4}\frac{M_\infty^{\ 4}}{\beta^2}\right)\cos 3\phi$$

Comparison of this expression with the right-hand side of equation (25b) shows that

$$\beta^3 D = -\frac{1}{4} - \frac{1}{8}\beta^2 + \frac{\gamma + 1}{32}\frac{M_\infty^{\ 4}}{\beta^2}\left[5 + 3\beta^2 + 3(\gamma + 1)\frac{M_\infty^{\ 2}}{\beta^2} - \frac{1}{2}(\gamma + 1)\frac{M_\infty^{\ 4}}{\beta^2}\right]$$

and

$$\beta^3 E = \frac{7}{24} + \frac{1}{8}\beta^2 - \frac{1}{24}\beta^4 + \frac{\gamma + 1}{32}\frac{M_\infty^4}{\beta^2}\left[1 - \frac{2}{3}\beta^2 + \frac{2}{9}(\gamma + 1)\frac{M_\infty^2}{\beta^2}\right]$$

Thus, to the third order of ϵ/β,

$$x = \phi - \frac{\epsilon}{\beta}e^{-\beta\psi}\sin\phi + \frac{1}{4}\left(\frac{\epsilon}{\beta}\right)^2\left[1 + \beta^2 - \frac{\gamma + 1}{4}\frac{M_\infty^4}{\beta^2}(2\beta\psi + 1)\right]e^{-2\beta\psi}\sin 2\phi +$$

$$\left(\frac{\epsilon}{\beta}\right)^3\left(\left\{\left[\frac{1}{4} + \frac{1}{8}\beta^2 - \frac{\gamma + 1}{32}\frac{M_\infty^4}{\beta^2}\left[5 + 3\beta^2 + 3(\gamma + 1)\frac{M_\infty^2}{\beta^2} - \right.\right.\right.\right.$$

$$\left.\left.\left.\frac{1}{2}(\gamma + 1)\frac{M_\infty^4}{\beta^2}\right]\right]\right\}e^{-\beta\psi}\sin\phi - \left(\frac{1}{3} + \frac{1}{8}\beta^2 - \frac{1}{12}\beta^4\right)e^{-3\beta\psi}\sin 3\phi +$$

$$\frac{\gamma + 1}{32}\frac{M_\infty^4}{\beta^2}e^{-3\beta\psi}\left\{\left[2 - \beta^2 + (\gamma + 1)\frac{M_\infty^2}{\beta^2} + \frac{1}{2}(\gamma + 1)\frac{M_\infty^4}{\beta^2}\beta\psi\right]\sin\phi - \right.$$

$$\left[3 - \frac{4}{3}\beta^2 + \frac{4}{9}(\gamma + 1)\frac{M_\infty^2}{\beta^2} + \frac{1}{2}(\gamma + 1)\frac{M_\infty^4}{\beta^2}\right]\sin 3\phi -$$

$$\left[4 + \frac{4}{3}(\gamma + 1)\frac{M_\infty^2}{\beta^2} + \frac{3}{2}(\gamma + 1)\frac{M_\infty^4}{\beta^2}\right]\beta\psi\sin 3\phi -$$

$$\left.\left.(\gamma + 1)\frac{M_\infty^4}{\beta^2}(\beta\psi)^2\sin 3\phi\right\}\right) \tag{26a}$$

and

$$
\begin{aligned}
\frac{1}{\beta} y = {} & \frac{1}{\beta} \psi + \frac{\epsilon}{\beta} e^{-\beta\psi} \cos \phi + \frac{1}{2}\left(\frac{\epsilon}{\beta}\right)^2 \left\{1 + \frac{\gamma + 1}{4} \frac{M_\infty{}^4}{\beta^2} - e^{-2\beta\psi}\left[\frac{\gamma + 1}{4} \frac{M_\infty{}^4}{\beta^2} + \right.\right. \\
& \left.\left.\left(1 - \frac{\gamma + 1}{4} \frac{M_\infty{}^4}{\beta^2} \beta\psi\right) \cos 2\phi\right]\right\} + \left(\frac{\epsilon}{\beta}\right)^3 \left(\left\{- \frac{1}{4} - \frac{1}{8} \beta^2 + \right.\right. \\
& \left.\frac{\gamma + 1}{32} \frac{M_\infty{}^4}{\beta^2}\left[5 + 3\beta^2 + 3(\gamma + 1) \frac{M_\infty{}^2}{\beta^2} - \frac{1}{2} (\gamma + 1) \frac{M_\infty{}^4}{\beta^2}\right]\right\} e^{-\beta\psi} \cos \phi + \\
& \left(\frac{1}{4} + \frac{1}{8} \beta^2\right) e^{-3\beta\psi} \cos 3\phi + \frac{\gamma + 1}{32} \frac{M_\infty{}^4}{\beta^2} e^{-3\beta\psi} \left\{-\left[4 + 3\beta^2 + \right.\right. \\
& \left. 3(\gamma + 1) \frac{M_\infty{}^2}{\beta^2} - \frac{1}{2} (\gamma + 1) \frac{M_\infty{}^4}{\beta^2} + \frac{3}{2} (\gamma + 1) \frac{M_\infty{}^4}{\beta^2} \beta\psi\right] \cos \phi - \\
& \cos 3\phi + \left[\frac{4}{3} + \frac{8}{3} \beta^2 + \frac{4}{3} (\gamma + 1) \frac{M_\infty{}^2}{\beta^2} + \frac{5}{6} (\gamma + 1) \frac{M_\infty{}^4}{\beta^2}\right] \beta\psi \cos 3\phi + \\
& \left.\left.(\gamma + 1) \frac{M_\infty{}^4}{\beta^2} (\beta\psi)^2 \cos 3\phi\right\}\right)
\end{aligned}
\tag{26b}
$$

For constant values of the stream function $\psi \geqq 0$, equations (26) comprise the parametric equations of the streamlines. The corresponding equations for incompressible flow are obtained by putting $M_\infty = 0$ or $\beta = 1$; that is,

$$x = \phi - \epsilon e^{-\psi}\sin\phi + \frac{1}{2}\epsilon^2 e^{-2\psi}\sin 2\phi + \frac{3}{8}\epsilon^3\left(e^{-\psi}\sin\phi - e^{-3\psi}\sin 3\phi\right) + \ldots \tag{27a}$$

and

$$y = \psi + \epsilon e^{-\psi}\cos\phi + \frac{1}{2}\epsilon^2\left(1 - e^{-2\psi}\cos 2\phi\right) - \frac{3}{8}\epsilon^3\left(e^{-\psi}\cos\phi - e^{-3\psi}\cos 3\phi\right) + \ldots \tag{27b}$$

As an example of the comparison of streamlines in incompressible and subsonic compressible flows, consider the case of $\epsilon = 0.64$, $M = 0.60$ or $\beta = 0.80$, and $\psi = 2.5$. The choice of 0.64 for ϵ would normally be too large if flow properties were to be calculated at the boundary itself with the aid of equations (26). Because of the presence of the exponential terms in these equations, however, a rather distant streamline such as $\psi = 2.5$ can be accurately determined. Thus, equations (26) become

$$x = \phi - 0.0495 \sin\phi + 0.0030 \sin 2\phi - 0.0008 \sin 3\phi + \ldots$$

and

$$y = 2.7865 + 0.0772 \cos\phi - 0.0036 \cos 2\phi + 0.0005 \cos 3\phi + \ldots$$

Equations (27), for a neighboring incompressible streamline given by $\psi = 2.6$, become

$$x = \phi - 0.0402 \sin\phi + 0.0011 \sin 2\phi + \ldots$$

and

$$y = 2.8048 + 0.0402 \cos \phi - 0.0011 \cos 2\phi + \ldots$$

A comparison of these two sets of equations shows at a glance the well-known general result that disturbances caused by the presence of a solid boundary die out more slowly for compressible flow than for incompressible flow.

CONCLUDING REMARKS

The use of the velocity potential ϕ and the stream function ψ as independent variables is not new in aeronautical literature. A. Thom and his students (particularly L. C. Woods) at the Oxford Engineering Laboratory have for years utilized incompressible ϕ and ψ as independent variables with the velocity vector components $\log \frac{1}{q}$ and θ (hodograph plane) as dependent variables. Numerical relaxation methods were especially developed for the ϕ,ψ-plane and employed in the solution of a number of diverse problems in both incompressible and compressible flows, including mixed subsonic and supersonic flow. In the present paper, the investigation is completely analytical and is confined to the physical-flow plane. In fact, the compressible velocity potential and stream function are thought of as presenting a set of orthogonal curvilinear coordinates associated with the shape of the solid boundary in the flow plane. The solution of a given flow problem is then expressed in the form of a pair of equations transforming the dependent rectangular Cartesian coordinates x and y into the independent orthogonal curvilinear coordinates ϕ and ψ.

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National Advisory Committee for Aeronautics,
Langley Field, Va., May 17, 1954.

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